The Ising model

1. Mean field Ising model in the microcanonical ensemble: Get the equation of the state for the Ising model on a d-dimensional lattice with coordination number z under the mean field approximation by directly evaluating the entropy $S = k_B \log \Omega(N, m)$ and writing the free energy as $A = U - TS$. Make the mean field approximation for the energy $U = \langle H \rangle$ and note that the average magnetization $m = N_+ - N_-$.

2. A generalization of the Ising model is the q-state Potts model, where one assume that the spin variable at a site can now take q different values, $\sigma_i = \{1, 2, 3, .., q\}$ and the Hamiltonian is given by
$$H_{Potts} = -J \sum_{<i,j>} \delta_{\sigma_i \sigma_j} - h \sum_i \delta_{\sigma_i,1}.$$
$< i, j >$ denote nearest neighbour interaction. By defining a suitable transformation between the Ising spin variable S_i and the Kronecker delta $\delta_{\sigma_i \sigma_j}$, show that the for $q = 2$, this Hamiltonian is equivalent to the Ising Hamiltonian.

3. Critical exponents for the 1D Ising model: Recall that we had argued in class that $T = 0$ is a true critical point for the 1D Ising model and in analogy with the formalism in higher dimensions, we can define the critical exponents α, β, γ, δ, ν, and η but with $t = \exp(-2J/k_B T)$.

   Show that $\alpha = 1$, $\beta = 0$, $\gamma = 1$, $\delta = \infty$, $\nu = 1$, and $\eta = 1$

   [Note that the exponents follow $2 - \alpha = \gamma = \nu$, which is a general result, always valid in any dimension and for any Hamiltonian!]

4. Critical exponents for the Ising model in the mean field approximation: Get the values α, β, γ, δ, ν, and η in the mean field approximation and verify (you may look up some text book!) that these values are the same as those for the van der Waals fluid.

5. Ising Model with 'antiferromagnetic exchange'. Consider the Ising model on a two-dimensional square lattice with nearest neighbour interaction but with the interaction term having opposite sign such that antiparallel allignment is favoured.
$$H = +|J| \sum_{<i,j>} S_i S_j - h \sum_{i=1}^{N} S_i$$
   (a) Show that for $h = 0$ the partition function and hence the free energy is the same as that for the usual Ising model. [Hint:This part is nicely solved in Goldenfeld, section 2.7.2 page 41!]

   (b) Solve the problem in the mean field approximation with $h = 0$.
   [Hint:You will end up with a pair of coupled transcendental equations. The order parameter is the staggered magnetization, i.e., the difference between the average magnetization between the two sublattices.]

6. Consider a one-dimensional Ising model but now with $S_i = 0, \pm 1$ and the external magnetic field being zero, i.e.,
$$H = \sum_{<i,j>} S_i S_j.$$
   Write down the expression for the transfer matrix. Use some software to evaluate the eigenvalues of this 3×3 matrix.

February 10, 2019
Instructor: Bhavtosh Bansal